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Polyharmonic Cardinal Splines

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Polyharmonic splines, sometimes called thin plate splines, are distributions which are annihilated by iterates of the Laplacian in the complement of a discrete set in Euclidean n-space and satisfy certain continuity conditions. The term cardinal is often used when the set of knots is a lattice. Here, in addition to developing certain basic properties of polyharmonic cardinal splines, it is shown that such splines interpolate numerical data on the lattice uniquely. © 1990 Academic Press, Inc.

1. INTRODUCTION

From one point of view, I. J. Schoenberg's theory of univariate cardinal splines of odd order, see [15], can be regarded as a development of certain properties of those functions, f, which satisfy

$$\frac{d^{2k}f}{dx^{2k}} = 0 \tag{1}$$

on the complement of the integer lattice Z and enjoy appropriate smoothness conditions on all of the real line R. A natural extension of these ideas to the multivariate case would be to consider those functions f which satisfy differential equations analogous to (1) on the complement of the integer lattice Z^n in R^n and enjoy certain regularity properties on all of R^n. In this development we consider the case where the differential operators

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are powers of the Laplacian. Many of the results in [15] have appropriate analogues in this case. In this paper we consider only the basic properties of such splines together with the problems of existence and uniqueness of cardinal interpolation.

The motivation for our work came from an attempt to obtain a "B spline like" basis for certain global interpolation schemes in R^n. More precisely, given a collection of points $x_1, ..., x_m$, in R^n, consider the linear subspace of functions, f, of the form

$$f(x) = \sum_{j=1}^{m} a_j \phi(x - x_j), \tag{2}$$

where ϕ is some fixed smooth function and $a_1, ..., a_m$ are arbitrary coefficients; call this subspace V_ϕ. Such functions are natural and simple candidates for interpolants of multivariate scattered data. In several interesting examples, such as $\phi(x) = |x|$ or $\phi(x) = \sqrt{1 + |x|^2}$, the function ϕ does not decay at infinity; see [7, 8] for these and other examples. As one may suspect, this causes various problems, both practical and theoretical. However, it is not difficult to see that, at least for the examples mentioned above, certain linear combinations of translates of ϕ, namely functions of form (2), decay at infinity rather quickly. It was hoped that such combinations would form a nice basis for V_ϕ analogous to that formed by the B splines in the classic univariate examples. In attempting to formulate a tractable theory we were led to consider the case where the set of points $x_1, ..., x_m$ becomes the integer lattice Z^n in R^n and the functions ϕ are fundamental solutions to certain powers of the Laplacian. This motivated us to examine Schoenberg's work [14, 15] more carefully and resulted in the development introduced here.

The idea of interpolating in terms of linear combinations of Green's functions to powers of the Laplacian is not new. Although the earliest published work devoted to the subject seems to be [10], it is quite clear that many mathematicians were aware of the idea and many of its consequences, either from the reproducing kernel Hilbert space viewpoint or from the transparent generalization of the variational aspect of univariate spline theory to the multivariate case; for instance, see [9]. For examples of more recent work see [6, 7, 8, 13]. Among the shortcomings of the obvious theory were the fact that such "splines" do not have a localized basis and the restriction to the case of the finite domain. In [6] Duchon developed a variational theory for interpolation to all of R^n involving a finite number of constraints which overcomes the second mentioned shortcoming in an elegant way and allows for interesting generalizations. Although our development concerns an infinite number of constraints and does not rely on the variational properties of splines, it may also be regarded as a certain generalization of [6].

While working on an early draft of our theory we discovered that the subject of "cardinal spline interpolation" was a booming business; for example, see [3, 4], and the references cited there. (In addition to references on multivariate cardinal splines, the extensive bibliography in [3] contains many references to work involving the Green's function type splines mentioned above.) However, although some of our results may have analogues in the works cited above, we felt that this development had enough novelty to warrant completion.

This paper is organized as follows. In Section 2 we give the definitions and derive the basic properties of k-harmonic splines. The cardinal interpolation problem for k-harmonic splines is taken up in Section 3, where it is shown that under certain circumstances this problem has a unique solution. The relationship of these splines to the multivariate analogue of the Whittaker cardinal series is indicated at the end of that section; this was prompted by a question raised by C. de Boor.

We use mathematical notation which is standard when dealing with multivariate functions. For example, the symbols μ and ν usually will denote multi-indices $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$, etc., $\mathscr{S}(R^n)$ and $\mathscr{S}'(R^n)$ denote the Schwartz space of rapidly decreasing functions and its dual, the space of tempered distributions. The introductory chapter of [11] contains a concise summary of this so-called multi-index notation and basic facts on distributions and Fourier transforms; other references which contain basic material used here concerning distributions, Fourier transforms, and several complex variables are [2, 5, 16]. For the Fourier transform we use a standard normalization which is slightly different from that used in [11], namely,

$$\hat{\phi}(\xi) = (2\pi)^{-n/2} \int e^{-i\langle \xi, x \rangle} \phi(x)\, dx$$

when ϕ is in $\mathscr{S}(R^n)$; here the integral is taken over all of R^n and $\hat{\phi}$ is the Fourier transform of ϕ. In what follows, integrals, as in the above case, are taken over R^n unless specifically denoted otherwise.

2. Definitions and Basic Properties

Recall that a function or distribution u is said to be *k-harmonic*, k a positive integer, if

$$\Delta^k u = 0 \tag{3}$$

on R^n. Here Δ is the usual Laplace operator defined by

$$\Delta u = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}$$

and, if k is greater than one, Δ^k denotes its kth iterate, $\Delta^k u = \Delta(\Delta^{k-1}u)$. Of course $\Delta^1 = \Delta$. A *polyharmonic* function is one which is k-harmonic for some positive integer k; for a treatise on the subject see [1].

The class $H_k(R^n)$ is the collection of all k-harmonic tempered distributions. In what follows the classes $H_k(R^n)$ play a role similar to that played by the polynomials, $\Pi_{2k-1}(R)$, in univariate spline theory. Indeed we have the following.

PROPOSITION 1. *The class $H_k(R^n)$ is a subspace of polynomials which contains $\Pi_{2k-1}(R^n)$.*

Proof. If u satisfies (3) then its Fourier transform, $\hat{u}$, satisfies $|\xi|^{2k}\,\hat{u}(\xi) = 0$ for all ξ in R^n. It follows that $\hat{u}$ is a distribution supported at the origin and thus must be a finite linear combination of the Dirac distribution and its derivatives. Hence u must be a polynomial. That $H_k(R^n)$ contains $\Pi_{2k-1}(R^n)$ follows from the fact that Δ^k is a homogeneous differential operator of order $2k$. ∎

For k a positive integer satisfying $2k \geqslant n+1$, we define $SH_k(R^n)$ to be the subspace of $\mathscr{S}'(R^n)$ whose elements f enjoy the properties

$$\begin{aligned} &\text{(i)} \quad f \text{ is in } C^{2k-n-1}(R^n) \\ &\text{(ii)} \quad \Delta^k f = 0 \text{ on } R^n \backslash Z^n. \end{aligned} \tag{4}$$

Here Z denotes the set of integers and Z^n denotes the integer lattice in R^n. Elements of Z^n are denoted by boldface symbols such as $\mathbf{j}$ and $\mathbf{m}$.

A *k-harmonic cardinal spline* is an element of $SH_k(R^n)$. We say that a function or distribution is a *polyharmonic cardinal spline* if it is in one of the classes $SH_k(R^n)$.

The reason for the condition $2k \geqslant n+1$ can be explained as follows. If $2k < n+1$ any distribution which is locally in L^∞ and satisfies condition (4)(ii) must be k-harmonic on R^n. Thus if $SH_k(R^n)$ is to consist of functions which satisfy (4)(ii) and for which pointwise evaluation on Z^n makes sense then the last observation implies that $SH_k(R^n)$ must in fact be the space $H_k(R^n)$. Like the space of polynomials in the case $n = 1$, the class $H_k(R^n)$ can only interpolate very restricted data on Z^n. Since we wish the space $SH_k(R^n)$ to be rich enough to interpolate a wide class of data on Z^n, we assume $2k \geqslant n+1$.

In what follows we always assume that $2k \geqslant n+1$. For the sake of clarity, however, we will remind the reader of this from time to time.

Fundamental solutions of (3) play an important role in the description of $SH_k(R^n)$. For future reference we define E_k to be the fundamental solution of (3) given by

$$E_k(x) = \begin{cases} c(n,k)|x|^{2k-n} & \text{if } n \text{ is odd} \\ c(n,k)|x|^{2k-n}\log|x| & \text{if } n \text{ is even,} \end{cases} \tag{5}$$

where $c(n, k)$ is a constant which depends only on n and k and is chosen so that $\Delta^k E_k(x) = \delta(x)$. Here $\delta(x)$ denotes the unit Dirac distribution at the origin. The Fourier transform of E_k is

$$\hat{E}_k(\xi) = (2\pi)^{-n/2}(-|\xi|^2)^{-k}. \tag{6}$$

The following propositions give some of the basic properties of the spaces $SH_k(R^n)$.

PROPOSITION 2. *If f is a tempered distribution then the following conditions are equivalent*:

(i) *f is in $SH_k(R^n)$.*

(ii) *f satisfies*

$$\Delta^k f(x) = \sum a_{\mathbf{j}}\delta(x - \mathbf{j}) \tag{7}$$

where the a's are constants and the sum is taken over all $\mathbf{j}$ in Z^n.

Proof. Suppose f is in $SH_k(R^n)$. To see that f satisfies (7) observe that in a sufficiently small neighborhood N of any point $\mathbf{j}$ we have

$$\Delta^k f(x) = \sum_{\nu} b_\nu D^\nu \delta(x - \mathbf{j}),$$

where the sum is taken over some finite set of multi-indices ν. Representation (7) will follow if we can show that $b_\nu = 0$ for $\nu \neq 0$. To see this let ϕ be any infinitely differentiable function with support in N such that $\phi(x) = 1$ in a neighborhood of $\mathbf{j}$. Then, since f is in $C^\infty(R^n \setminus Z_n)$,

$$\Delta^k(\phi f) = \sum_{\nu} b_\nu D^\nu \delta(x - \mathbf{j}) + \psi,$$

where ψ is an infinitely differentiable function with support in N. Now, if E is any fundamental solution of (3) then

$$\phi f = E * (\Delta^k \phi f) = \sum_{\nu} b_\nu D^\nu E(x - \mathbf{j}) + E * \psi,$$

where $E * \psi$ is infinitely differentiable on R^n. The fact that $b_\nu = 0$ for $\nu \neq 0$

follows from the last equation, the fact that ϕf is in $C^{2k-n-1}(R^n)$, and the behavior of $D^\nu E(x-\mathbf{j})$ in a neighborhood of $\mathbf{j}$.

To see that (7) implies that f is in $C^{2k-n-1}(R^n)$ observe that in any sufficiently small neighborhood N of $\mathbf{j}$ $f(x)-a_{\mathbf{j}}E(x-\mathbf{j})$ is equal to a k-harmonic function and therefore $f(x)-a_{\mathbf{j}}E(x-\mathbf{j})$ is in $C^\infty(N)$. Note that $E(x-\mathbf{j})$ is in $C^{2k-n-1}(R^n)$ and hence it follows that f is in $C^{2k-n-1}(N)$. The desired conclusion is now an easy consequence of this fact. ∎

In what follows we say that a sequence $a_{\mathbf{j}}$, $\mathbf{j}$ in Z^n, is of polynomial growth if there are constants c and p such that $|a_{\mathbf{j}}| \leqslant c(1+|\mathbf{j}|)^p$ for all $\mathbf{j}$. Similarly a locally bounded function f is said to be of polynomial growth if $|f(x)| \leqslant c(1+|x|)^p$ for all x in R^n. The following observation is an easy consequence of the definitions.

Proposition 3. *If f is in $SH_k(R^n)$ then the coefficients in representation (7) are unique and are of polynomial growth. Furthermore, if f_1 and f_2 are in $SH_k(R^n)$ and $\Delta^k f_1 = \Delta^k f_2$ then $f_1 - f_2$ is in $H_k(R^n)$.*

The proof of the following proposition is somewhat technical and perhaps disruptive to the flow of main ideas of this section. We include it for completeness.

Proposition 4. *If f is in $SH_k(R^n)$ then f is of polynomial growth.*

Proof. For ϕ in $\mathscr{S}(R^n)$ let

$$|\phi|_{\mu,\nu} = \sup|x^\mu D^\nu \phi(x)|,$$

where the supremum is taken over all x in R^n. Since f is in $\mathscr{S}'(R^n)$ there are constants M, N, and C_1 so that

$$|\langle f, \phi\rangle| \leqslant C_1 \sum |\phi|_{\mu,\nu}, \tag{8}$$

where the sum is taken over all multi-indices μ and ν such that $0 \leqslant |\mu| \leqslant M$ and $0 \leqslant |\nu| \leqslant N$. Since f is in $SH_k(R^n)$ there are constants M_1 and C_2 such that

$$|\langle \Delta^k f, \phi\rangle| \leqslant C_2 \sum |\phi|_{\mu,0}, \tag{9}$$

where the sum is taken over all multi-indices μ such that $0 \leqslant |\mu| \leqslant M_1$.

Observe that $\langle f, \phi\rangle = \langle f, \phi_1\rangle + \langle(-\Delta)^k f, \phi_1\rangle$ and by induction

$$\langle f, \phi\rangle = \langle f, \phi_m\rangle + \sum_{j=1}^{m} \langle(-\Delta)^k f, \phi_j\rangle, \tag{10}$$

where ϕ_j is defined by the formula for its Fourier transform

$$\hat{\phi}_j(\xi) = (1 + |\xi|^{2k})^{-j} \hat{\phi}(\xi). \tag{11}$$

Formula (11) together with properties of the Fourier transform imply that

$$x^\mu \phi_j(x) = \sum_{\nu \leqslant \mu} g_\nu * \psi_\nu(x), \tag{12}$$

where ψ_ν is defined by $\psi_\nu(x) = x^\nu \phi(x)$ and the g_ν are bounded continuous functions. Hence there is a constant C_3 such that

$$|\phi_j|_{\mu,0} \leqslant C_3 \sum_{\nu \leqslant \mu} \int |x^\nu \phi(x)| \, dx. \tag{13}$$

Similar reasoning shows that if m satisfies $2k(m-1) > |\nu|$ then

$$|\phi_m|_{\mu,\nu} \leqslant C_4 \sum_{\kappa \leqslant \mu} \int |x^\kappa \phi(x)| \, dx, \tag{14}$$

where C_4 is a constant independent of ϕ.

Formula (10) together with estimates (8), (9), (13), and (14) imply that there are positive constants C and α, independent of ϕ, such that

$$|\langle f, \phi \rangle| \leqslant C \int (1 + |x|)^\alpha \, |\phi(x)| \, dx \tag{15}$$

for all ϕ in $\mathscr{S}(R^n)$. Estimate (15) together with Riesz representation implies the desired result. ∎

This last proposition, when combined with the fact that continuous functions of polynomial growth are distributions in $\mathscr{S}'(R^n)$, shows that $SH_k(R^n)$ could be defined, without any loss of generality, as the class of continuous functions which satisfy (4) and are of polynomial growth.

In the case $n = 1$, $SH_k(R^n)$ is the subspace of S_m, $m = 2k - 1$, consisting of functions of polynomial growth. The space S_m is the space of piecewise polynomial functions defined by Schoenberg; see [15]. The reason we restrict our attention to $\mathscr{S}'(R^n)$ is because our development relies on the use of the Fourier transform. Thus in spirit this development is similar to that in Schoenberg's earlier work [14].

If α is a real parameter we define $SH_k^\alpha(R^n)$ to be that subspace of $SH_k(R^n)$ consisting of functions f which satisfy $|f(x)| \leqslant c(1 + |x|)^\alpha$ for some constant c. As an immediate consequence of Proposition 4 we have the following.

COROLLARY 1. *$SH_k(R^n) = \bigcup SH_k^\alpha(R^n)$, where the union is taken over all real α.*

To complete our list of basic properties of $SH_k(R^n)$ we include the following.

PROPOSITION 5. *Given a sequence $\{a_{\mathbf{j}}\}$, $\mathbf{j}$ in Z^n, which is of polynomial growth, there is an f in $SH_k(R^n)$ such that*

$$\Delta^k f(x) = \sum a_{\mathbf{j}} \delta(x - \mathbf{j}). \tag{16}$$

Proof. Recall that $\Delta^k u = v$ has a solution u in $\mathcal{S}'(R^n)$ whenever v is in $\mathcal{S}'(R^n)$; for example, see [12]. The hypothesis implies that the right-hand side of (16) is a tempered distribution and thus there is a tempered distribution f such that (16) holds. That f is in $SH_k(R^n)$ now follows from Proposition 2. ∎

PROPOSITION 6. *$SH_k(R^n)$ is a closed subspace of $\mathcal{S}'(R^n)$.*

Proof. The mapping $f \to \Delta^k f$ is continuous as an operator from $\mathcal{S}'(R^n)$ to $\mathcal{S}'(R^n)$, which means that the inverse image of a closed set is closed under this mapping. Now, the set of those elements in $\mathcal{S}'(R^n)$ consisting of Radon measures supported on Z^n is a closed subspace of $\mathcal{S}'(R^n)$, and, since $SH_k(R^n)$ is the inverse image of this subspace under the mapping described above, the desired result follows. ∎

3. CARDINAL SPLINE INTERPOLATION

The problem of *cardinal interpolation for k-harmonic splines* is the following:

> Given a sequence of real or complex numbers, $v = \{v_{\mathbf{j}}\}$, $\mathbf{j}$ in Z^n,
>
> find an element f in $SH_k(R^n)$ such that $f(\mathbf{j}) = v_{\mathbf{j}}$ for all $\mathbf{j}$.

This, of course, is a simple generalization of the standard problem in the univariate case; see [15, p. 33].

Since elements of $SH_k(R^n)$ are of polynomial growth it is clear that in order for this problem to have a solution a necessary requirement on the sequence v is that it also be of polynomial growth. This requirement is also sufficient, as we shall show. We begin by first considering the fundamental functions of interpolation.

The function L_k is defined by the formula for its Fourier transform

$$\hat{L}_k(\xi) = (2\pi)^{-n/2} \frac{|\xi|^{-2k}}{\sum_{\mathbf{j} \in Z^n} |\xi - 2\pi\mathbf{j}|^{-2k}}. \tag{17}$$

If k is an integer such that $2k \geq n+1$ then $L_k(x)$ is well defined as an absolutely convergent integral. This function is the *fundamental function of interpolation* for k-harmonic splines. The reason for this terminology is the fact that $L_k(\mathbf{j}) = \delta_{0\mathbf{j}}$, where $\delta_{0\mathbf{j}}$ is the Kronecker delta. This and other properties of L_k and some of the consequences for the interpolation problem are stated in Propositions 7 and 8. Before turning to these propositions, however, we need to consider certain technical properties and lemmas concerning L_k and the related periodic distribution, $\hat{\Phi}_k$, defined by

$$\hat{\Phi}_k(\xi) = (-|\xi|^2)^k \hat{L}_k(\xi). \tag{18}$$

Given a subset $\mathscr{A}$ of the real line R and a positive number ε, $\mathscr{A}_\varepsilon$ is the subset of the complex plane defined by

$$\mathscr{A}_\varepsilon = \{\tau \in C : \Re\tau \in \mathscr{A} \text{ and } -\varepsilon < \Im\tau < \varepsilon\},$$

where $\Re\tau$ and $\Im\tau$ are the real and imaginary parts of τ, respectively. Similarly $\mathscr{A}^n_\varepsilon = \mathscr{A}_\varepsilon \times \cdots \times \mathscr{A}_\varepsilon$ is a subset of C^n. The symbol Q denotes the interval $-\pi < \rho \leq \pi$ and Q^n denotes the cube

$$Q^n = \{\xi = (\xi_1, ..., \xi_n) : -\pi < \xi_j \leq \pi, j = 1, ..., n\}.$$

LEMMA 1. *The function $\hat{\Phi}_k$ and $\hat{L}_k$ have extensions which are analytic in a tube R^n_ε, for some $\varepsilon > 0$.*

Proof. For ξ and η in R^n, let

$$\zeta = (\zeta_1, ..., \zeta_n) = (\xi_1 + i\eta_1, ..., \xi_n + i\eta_n) = \xi + i\eta$$

denote a point in complex n-space, C^n, and put

$$q(\zeta) = -\sum_{m=1}^{n} \zeta_m^2.$$

Observe that

$$(2\pi)^{n/2} [\hat{\Phi}_k(\xi)]^{-1} = \{1 + [q(\xi)]^k F(\xi)\}[q(\xi)]^{-k}, \tag{19}$$

where

$$F(\zeta) = \sum_{\mathbf{j} \in Z^n \setminus \{0\}} q(\zeta - 2\pi\mathbf{j})^{-k}.$$

Choose ε small enough so that $q(\zeta) \neq 0$ for all ζ in $R_\varepsilon^n \backslash Q_\varepsilon^n$. Then it is readily checked that F is analytic in Q_ε^n. For ξ in Q^n, $[q(\xi)]^k F(\xi) \geqslant 0$. Thus, by reducing ε if necessary, we may assume that $1 + [q(\zeta)]^k F(\zeta)$ has no zeros in Q_ε^n. Now, from (19), it follows that $\hat{\Phi}_k$ extends analytically to Q_ε^n and hence, by periodicity, it extends analytically to R_ε^n.

The analytic extension of $\hat{L}_k$ is given by $\hat{\Phi}_k(\zeta)[q(\zeta)]^{-k}$. From (19) it is evident that this extension is analytic in Q_ε^n. Analyticity on the rest of R_ε^n is clear from the fact that $q(\zeta) \neq 0$ there. ∎

LEMMA 2. *The Fourier transform of $\hat{\Phi}_k$ is a sum of constant multiples of translates of the delta function. More precisely, $\hat{\hat{\Phi}}_k = \Phi_k$ and*

$$\Phi_k(x) = \sum_{\mathbf{j} \in Z^n} a_\mathbf{j} \delta(x - \mathbf{j}), \tag{20}$$

where the $a_\mathbf{j}$'s depend on k and n. Furthermore, there are positive constants, C and c, which are independent of $\mathbf{j}$ such that

$$|a_\mathbf{j}| \leqslant C \exp(-c|\mathbf{j}|) \tag{21}$$

for all $\mathbf{j}$.

Proof. The periodicity of $\hat{\Phi}_k$ implies that Φ_k satisfies (20) with

$$a_\mathbf{j} = (2\pi)^{-2/2} \int_{Q^n} e^{i\langle \mathbf{j}, \xi \rangle} \hat{\Phi}_k(\xi)\, d\xi. \tag{22}$$

Now, by virtue of analyticity of $\hat{\Phi}_k$ the set Q^n in (22) can be replaced by $\{\zeta : \Re\zeta \in Q^n \text{ and } \Im\zeta = \gamma\}$, where $\gamma = (\gamma_1, ..., \gamma_n)$, γ_m are constants, $|\gamma_m| = \varepsilon/2$, $m = 1, ..., n$, and the sign of γ_m is chosen so that $\langle \mathbf{j}, \gamma \rangle > 0$ whenever $\mathbf{j} \neq 0$; here ε is the same as that in Lemma 1. Upon doing this, a routine estimate of the resulting integral gives (21). ∎

PROPOSITION 7. *Let L_k be defined by the formula for its Fourier transform* (17), *where k is an integer which satisfies $2k \geqslant n + 1$. Then L_k has the following properties*:

(i) *L_k is a k-harmonic cardinal spline.*

(ii) *For all $\mathbf{j}$ in Z^n*

$$L_k(\mathbf{j}) = \begin{cases} 1 & \text{if } \mathbf{j} = 0 \\ 0 & \text{if } \mathbf{j} \neq 0. \end{cases} \tag{23}$$

(iii) *There are positive constants A and a, depending on n and k but independent of x, such that*

$$|L_k(x)| \leqslant A \exp(-a|x|) \tag{24}$$

for all x in R^n.

(iv) *L_k has the following representations in terms of E_k:*

$$L_k(x) = \sum_{\mathbf{j} \in Z^n} a_{\mathbf{j}} E_k(x - \mathbf{j}) = \Phi_k * E_k(x), \tag{25}$$

where Φ_k is the function whose Fourier transform is defined by (18) and the $a_{\mathbf{j}}$'s are the constants defined in Lemma 2. The series converges absolutely and uniformly on all compact subsets of R^n.

Proof. (i) This follows from (20) and the fact that $\Delta^k L_k = \Phi_k$.

(ii) Observe that $\sum_{\mathbf{j}} \hat{L}_k(\xi - 2\pi\mathbf{j}) = (2\pi)^{-n/2}$, and write

$$L_k(\mathbf{j}) = (2\pi)^{-n/2} \int \hat{L}_k(\xi)\, e^{i\langle \mathbf{j}, \xi\rangle} d\xi = (2\pi)^{-n/2} \sum_{\mathbf{m} \in Z^n} \int_{Q^n} \hat{L}_k(\xi - 2\pi\mathbf{m})\, e^{i\langle \mathbf{j}, \xi\rangle} d\xi$$

$$= (2\pi)^{-n/2} \int_{Q^n} \sum_{\mathbf{m} \in Z^n} \hat{L}_k(\xi - 2\pi\mathbf{m})\, e^{i\langle \mathbf{j}, \xi\rangle} d\xi = \frac{1}{(2\pi)^n} \int_{Q^n} e^{i\langle \mathbf{j}, \xi\rangle} d\xi.$$

The desired result is now an immediate consequence of the last formula.

(iii) The proof of this estimate is analogous to the proof of (21) in Lemma 2.

(iv) This is a transparent consequence of the definitions and Lemma 2. ∎

For later reference, we define $\mathscr{Y}^\alpha$, α real, to be the collection of those sequences $v = \{v_{\mathbf{j}}\}$, $\mathbf{j}$ in Z^n, which satisfy

$$|v_{\mathbf{j}}| \leqslant c(1 + |\mathbf{j}|)^\alpha \tag{26}$$

for some constant c. Note that these classes are analogous to the ones in [15, p. 34]; they will also be used here in a similar fashion.

PROPOSITION 8. *Suppose $v = \{v_{\mathbf{j}}\}$, $\mathbf{j}$ in Z^n, is a sequence of polynomial growth and f_v is the function defined by*

$$f_v(x) = \sum_{\mathbf{j} \in Z^n} v_{\mathbf{j}} L_k(x - \mathbf{j}). \tag{27}$$

Then the following are true:

(i) *The expansion (27) converges absolutely and uniformly in every compact subset of R^n.*

(ii) *The function f_v is a k-harmonic spline and $f_v(\mathbf{j}) = v_{\mathbf{j}}$ for all $\mathbf{j}$.*

(iii) *If v is in $\mathscr{Y}^\alpha$ then f_v is in $SH_k^\alpha(R^n)$.*

(iv) *If $v_{\mathbf{j}} = P(\mathbf{j})$ for all $\mathbf{j}$, where $P(x)$ is a k-harmonic polynomial, then $f_v(x) = P(x)$ for all x in R^n.*

Proof. Items (i)–(iii) are immediate consequences of Proposition 7.

(iv) Suppose P is a k-harmonic polynomial and f is defined by

$$f(x) = \sum_{\mathbf{j} \in Z^n} P(\mathbf{j})\, L_k(x - \mathbf{j}).$$

Recall that any locally integrable function g of polynomial growth is in $\mathscr{S}'(R^n)$; furthermore for any ϕ in $\mathscr{S}(R^n)$ we have

$$\langle g, \phi \rangle = \int g(x)\, \phi(x)\, dx \qquad \text{and} \qquad g * \phi(x) = \int g(y)\, \phi(x - y)\, dy.$$

To prove (iv) of this proposition it suffices to show that

$$\langle f, \phi \rangle = \langle P, \phi \rangle \tag{28}$$

for all ϕ in $\mathscr{S}(R^n)$.

To see (28) observe that

$$\langle f, \phi \rangle = \sum_{\mathbf{j} \in Z^n} P(\mathbf{j})\, L_k * \phi(\mathbf{j}) = (2\pi)^{n/2} \sum_{\mathbf{j} \in Z^n} [P(iD)(\hat{L}_k(\xi)\, \phi(\xi))]_{\xi = 2\pi \mathbf{j}}, \tag{29}$$

where the last equality follows by virtue of the Poisson summation formula. To evaluate the expression on the extreme right in (29) set

$$A_{\mathbf{j}} = [P(iD)(\hat{L}_k(\xi)\, \hat{\phi}(\xi))]_{\xi = 2\pi \mathbf{j}}$$

and observe that for each $\mathbf{j}$ in Z^n, $\hat{L}_k$ may be expressed as

$$(2\pi)^{n/2}\, \hat{L}_k(\xi) = \begin{cases} 1 + |\xi|^{2k}\, \psi_0(\xi) & \text{if } \mathbf{j} = 0 \\ |\xi - 2\pi \mathbf{j}|^{2k}\, \psi_{\mathbf{j}}(\xi) & \text{otherwise,} \end{cases} \tag{30}$$

where the $\psi_{\mathbf{j}}$'s are smooth functions such that $\hat{\phi}_{\mathbf{j}}(\xi) = \psi_{\mathbf{j}}(\xi)\, \phi(\xi)$ is a test function in $\mathscr{S}(R^n)$ for each $\mathbf{j}$. In view of (30) we may write

$$(2\pi)^{n/2}\, A_{\mathbf{j}} = \begin{cases} \int P(x)[\phi(x) + (-\Delta)^k\, \phi_0(x)]\, dx & \text{if } \mathbf{j} = 0 \\ \int P(x)(-\Delta)^k\, [e^{-i\langle \mathbf{j}, x \rangle} \phi_{\mathbf{j}}(x)]\, dx & \text{otherwise.} \end{cases} \tag{31}$$

Now, integrating by parts and using the fact that $\Delta^k P = 0$ results in

$$\int P(x)(-\Delta)^k\, [e^{-i\langle \mathbf{j}, x \rangle} \phi_{\mathbf{j}}(x)]\, dx = \int [(-\Delta)^k\, P(x)]\, e^{-i\langle \mathbf{j}, x \rangle} \phi_{\mathbf{j}}(x)\, dx = 0 \tag{32}$$

when $\mathbf{j} \neq 0$. Similar reasoning shows that

$$(2\pi)^{n/2}\, A_0 = \int P(x)\, \phi(x)\, dx. \tag{33}$$

Formulas (29), (30), (31), (32), and (33) imply (28), which is the desired result. ∎

It should now be clear why L_k is called the fundamental function of interpolation for k-harmonic splines. Note that item (ii) of the previous proposition may be restated as follows.

COROLLARY 2. *If the data v are of polynomial growth then the cardinal interpolation problem for k-harmonic splines has a solution which has a unique representation in terms of fundamental functions of interpolation. The solution is given by*

$$f_v(x) = \sum_{\mathbf{j} \in Z^n} v_{\mathbf{j}} L_k(x - \mathbf{j}). \tag{34}$$

We now take up the question of uniqueness by first considering the following technical lemma.

LEMMA 3. *If f is in $SH_k(R^n)$ and f is Z^n periodic then f is a constant.*

Proof. The hypothesis implies that

$$\hat{f}(\xi) = \sum c_{\mathbf{j}} \delta(\xi - 2\pi\mathbf{j}) \qquad \text{and} \qquad \Delta^k f(x) = a \sum \delta(x - \mathbf{j}),$$

where both sums are taken over all $\mathbf{j}$ in Z^n; a and the $c_{\mathbf{j}}$'s are constants. These formulas together with Poisson summation imply

$$(-|2\pi\mathbf{j}|^2)^k\, c_{\mathbf{j}} = a \tag{35}$$

for all $\mathbf{j}$ in Z^n. The equation corresponding to $\mathbf{j} = 0$ in (35) implies that a is 0; the rest of the equations in system (35) imply that $c_{\mathbf{j}} = 0$ whenever $\mathbf{j} \neq 0$. Thus $f(x) = (2\pi)^{-n/2} C_0$, which is the desired result. ∎

PROPOSITION 9. *If f is in $SH_k(R^n)$ and $f(\mathbf{j}) = 0$ for all $\mathbf{j}$ in Z^n then f is identically* 0.

Proof. Recall that

$$\Delta^k f(x) = \sum_{\mathbf{j} \in Z^n} c_{\mathbf{j}} \delta(x - \mathbf{j}). \tag{36}$$

Now, if $c_{\mathbf{j}} = 0$ for all $\mathbf{j}$, f must be a k-harmonic polynomial. This together with the fact that $f(\mathbf{j}) = 0$ for all $\mathbf{j}$ implies the desired result.

Thus to complete the proof it suffices to show that $c_{\mathbf{j}} = 0$ for all $\mathbf{j}$.

To see this, write

$$g(x) = \sum_{\mathbf{j} \in Z^n} c_{\mathbf{j}} L_k(x - \mathbf{j}),$$

and observe that

$$\Delta^k(g - \Phi_k * f) = 0, \tag{37}$$

where Φ_k is the distribution defined by Lemma 2. From (37) it follows that $g - \Phi_k * f$ is a k-harmonic polynomial P. Since $\Phi_k * f(\mathbf{j}) = 0$ for all $\mathbf{j}$ it follows that $g(\mathbf{j}) = P(\mathbf{j}) = c_{\mathbf{j}}$.

Now, if $P \neq 0$, there is a finite difference operator, T, such that

$$TP(x) = \sum_{\mathbf{j} \in \mathscr{F}} b_{\mathbf{j}} P(x - \mathbf{j}) = B, \tag{38}$$

where the sum is taken over a finite subset $\mathscr{F}$ of Z^n and B is a non-zero constant. Write

$$\Delta^k Tf(x) = T\Delta^k f(x) = \sum_{\mathbf{j} \in Z^n} TP(\mathbf{j})\, \delta(x - \mathbf{j}) = B \sum_{\mathbf{j} \in Z^n} \delta(x - \mathbf{j}) \tag{39}$$

and observe that (39) means that $Tf(x - \mathbf{j}) - Tf(x)$ is a k-harmonic polynomial for each $\mathbf{j}$. This, together with the fact that $Tf(\mathbf{m} - \mathbf{j}) - Tf(\mathbf{m}) = 0$ for all $\mathbf{m}$ and $\mathbf{j}$ implies that Tf is Z^n periodic. Now, by virtue of Lemma 3, Tf is a constant and hence

$$\Delta^k Tf = 0, \tag{40}$$

Formulas (39) and (40) contradict the fact that $B \neq 0$ and thus imply the desired result. ∎

An immediate consequence of Proposition 9 of course is the fact that any solution for the problem of cardinal interpolation for k-harmonic splines is unique. We summarize our results concerning the interpolation problem as follows:

THEOREM 1. *If $v = \{v_{\mathbf{j}}\}$, $\mathbf{j}$ in Z^n, is a sequence of polynomial growth then there is a unique k-harmonic spline f such that $f(\mathbf{j}) = v_{\mathbf{j}}$ for all $\mathbf{j}$. Furthermore, if v is in $\mathscr{Y}^\alpha$ then f is in $SH_k^\alpha(R^n)$.*

Note that the converse of the theorem above follows immediately from the definitions and results in Section 2.

THEOREM 2. *Every k-harmonic spline f has a unique representation in terms of translates of L_k, namely*

$$f(x) = \sum_{\mathbf{j} \in Z^n} f(\mathbf{j})\, L_k(x - \mathbf{j}).$$

We conclude this paper with a result that may justify the use of the adjective "cardinal" when referring to these splines. For the univariate case see [15].

PROPOSITION 10. *For* $x = (x_1, ..., x_n)$

$$\lim_{k \to \infty} L_k(x) = \prod_{j=1}^{n} \frac{\sin \pi x_j}{\pi x_j} \tag{41}$$

uniformly in x *on* R^n.

Proof. First observe that the formula for $\hat{L}_k$ may be rewritten as

$$\hat{L}_k(\xi) = (2\pi)^{-n/2} \left\{ 1 + \sum_{\mathbf{j} \neq 0} \frac{|\xi|^{2k}}{|\xi - 2\pi \mathbf{j}|^{2k}} \right\}^{-1}. \tag{42}$$

Now, for $\mathbf{j} \neq 0$ and ξ in the interior of Q^n, $|\xi - 2\pi\mathbf{j}| > |\xi|$. Hence, for such ξ, it is clear from (42) that

$$\lim_{k \to \infty} \hat{L}_k(\xi) = (2\pi)^{-n/2}. \tag{43}$$

To see what happens for general ξ's write, by virtue of the periodicity of $\hat{\Phi}_k$,

$$\hat{L}_k(\xi) = (-|\xi|^2)^{-k} \hat{\Phi}_k(\xi - 2\pi\mathbf{j}) = \left\{ \frac{|\xi - 2\pi\mathbf{j}|}{|\xi|} \right\}^{2k} \hat{L}_k(\xi - 2\pi\mathbf{j}). \tag{44}$$

From (44) it is clear that for ξ in the complement of the closure of Q^n

$$\lim_{k \to \infty} \hat{L}_k(\xi) = 0. \tag{45}$$

Again using (44) together with routine estimates shows that, whenever $2k \geq n+1$, $\hat{L}_k$ is dominated by an integrable function, independent of k. This together with (43), (45), and the dominated convergence theorem implies that $(2\pi)^{n/2} \hat{L}_k$ converges to the characteristic (indicator) function of Q^n in $L^1(R^n)$ and hence, the desired result follows by taking Fourier transforms. ∎

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